CALCULATION OF VISCOUS AND PLASTIC PROPERTIES BY THE SOLUTION OF WAVE PROBLEMS

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Models of elastoplastic media are applied to soils and rocks [1, 2]. In conformity with experimental data [3-5] a model of soils and rocks as a viscoplastic medium has been proposed [6]. Below we give a solution, based on this model, of the problem on the propagation of a plane one-dimensional wave. As the basis of computer programs we propose a finite-difference representation of the equations of motion of a continuous medium in Lagrange coordinates and the differential equations governing the behavior of the medium. A "direct calculation" procedure with pseudoviscosity is applied. It is shown that the damping of plane waves is connected with two energy-dissipating mechanisms, determined by the viscous and plastic properties of the medium. The washing out of a discontinuity can occur in the absence of a segment of the dynamical compression curve that is concave to the strain axis. Under certain conditions the maximum strain is attained during the phase of decreasing stress. These results agree with the experimental data [3].

1. Soil is regarded as a medium consisting of solid mineral grains, cemented together by salt films and aqueous films. Under shock loading a strain ε_1 appears instantaneously, due to the compression of the material of the films and of the material of the grains themselves, and due also to the displacement of the grains that are in the least stable positions. Values of this strain lie on the diagram of dynamical compression of the medium

$$\sigma = E_D \varepsilon_1 \tag{1.1}$$

The strain ε_1 is assumed to be partially reversible – upon a decrease in the loading the volume is restored on account of the unloading of the material of the films and the grains. With a decrease in the loading from σ_*

$$\sigma - \sigma_* = E_R (\varepsilon_1 - \varepsilon_*), \quad \varepsilon_* = \sigma_* / E_D, \quad E_R \geqslant E_D \tag{1.2}$$

Under the action of the loading, a strain ε_2 occurs after a finite time, connected with the shifting and repacking of the grains. The magnitude of ε_2 is determined by the expression

$$\sigma = E_2 \varepsilon_2 + \eta \dot{\varepsilon}_2 \tag{1.3}$$

where η is the viscosity coefficient of the medium.

The limiting value of the overall deformation ε of the soil with $\sigma = \text{const}$ and $t \to \infty$ lies on the static compression diagram

$$\varepsilon = \varepsilon_1 + \varepsilon_2, \quad \sigma = E_S \varepsilon \tag{1.4}$$

$$E_D^{-1} + E_2^{-1} = E_S^{-1} \tag{1.5}$$

The magnitude of E_2 is determined from experimental values of E_D and E_S.

The repacking strain ε_2 is assumed to be irreversible, since the air that is contained in the pores cannot overcome the force of friction between the particles and return them to their original positions when the loading is removed.

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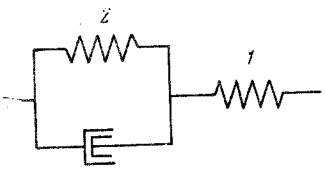

Fig. 1

A schematic drawing of an element is given in Fig. 1. If the strain of the dash pot and the springs 1 and 2 are reversible this is a viscoelastic model; with irreversibility of the strains of spring 2 and the dash pot we have a viscoplastic model. Within the framework of the viscoplastic model the strain of spring 1 can be completely or partially reversible.

We shall discuss the equation that determines the behavior of an element of the medium. At the shock front

$$\varepsilon = \varepsilon_1 = \sigma_* / E_D \tag{1.6}$$

For a continuous increase in the stress we obtain from (1.3) and (1.6)

$$\dot{\varepsilon} + \mu\varepsilon = \dot{\sigma} / E_D + \mu\dot{\sigma} / E_S, \quad \mu = E_2 / \eta \tag{1.7}$$

A similar equation was obtained earlier by A. Yu. Ishlinskii [7].

For a viscoelastic medium Eq. (1.7) is also valid when the stress decreases.

When the stress decreases in a viscoplastic medium, ε_1 decreases according to the equation

$$\varepsilon_1 = \varepsilon_* + (\sigma - \sigma_*) / E_R \tag{1.8}$$

where ε_* and σ_* are the maximum values of ε_1 and σ, attained in the element of the medium, respectively.

The behavior of the medium is determined by an equation that follows from (1.3) and (1.8)

$$\dot{\varepsilon} + \mu\varepsilon = \dot{\sigma}E_R^{-1} + \mu\sigma(E_S^{-1} - E_D^{-1} + E_R^{-1}) + \mu\sigma_*(E_D^{-1} - E_R^{-1}) \tag{1.9}$$

After the maximum strain ε is attained, it is assumed that ε_2=const, while ε_1 decreases according to (1.8). The behavior of the medium is determined by the equation

$$E_R\dot{\varepsilon} = \dot{\sigma} \tag{1.10}$$

We shall employ Lagrange variables: the mass h, and the time t. The wave is caused by a loading acting on the free surface of a half-space. On this section, that is assumed to move beyond the initial section, the loading increases by a jump to the value σ_m at t=0, and thereafter varies according to the equations

$$\sigma = \sigma_m(1 - t/\theta), \quad 0 \leqslant t \leqslant \theta, \quad \sigma = 0, \quad \theta \leqslant t \tag{1.11}$$

A shock front propagates from the initial section with the velocity

$$A = (E_D\rho_0)^{1/2}$$

where ρ_0 is the initial density of the medium.

In front of the shock front

$$\sigma = 0, \quad u = 0, \quad \rho = \rho_0$$

The equation of the trajectory of the front in the h, t plane is

$$h = At$$

The flow behind the front is determined by the fundamental equations of motion

$$\rho_0\frac{\partial u}{\partial h} - \frac{\partial \varepsilon}{\partial t} = 0, \qquad \frac{\partial u}{\partial t} - \frac{\partial \sigma}{\partial h} = 0 \tag{1.12}$$

This system is closed by equations determining the behavior of the medium, which are taken as those determined by the instantaneous state of the medium in the form (1.6), (1.7), (1.9), and (1.10).

The boundary conditions are: on the initial section $\sigma = \sigma_m(1-t/\theta)$ and on the line h=At we have $\sigma_* = Au_*$. The asterisks indicate that the corresponding quantities are taken for a particle with the coordinate h at the time t=h/A. We convert to the dimensionless quantities

$$\tau = \mu t, \quad x = \mu h / A, \quad \sigma^\circ = \sigma / \sigma_m, \quad u^\circ = u / u_m, \quad \varepsilon^\circ = \varepsilon / \varepsilon_m$$
$$u_m = -\sigma_m / A, \quad \varepsilon_m = \sigma_m / E_D \tag{1.13}$$

and we use the notations

$$\gamma = E_D / E_S, \quad \beta = E_D / E_R, \quad \lambda = \mu\theta$$

In terms of the new variables the equations of motion are

$$\frac{\partial u^\circ}{\partial \tau} + \frac{\partial \sigma^\circ}{\partial x} = 0, \qquad \frac{\partial u^\circ}{\partial x} + \frac{\partial \varepsilon^\circ}{\partial \tau} = 0 \tag{1.14}$$

The equations determining the behavior of the medium are

$$\begin{gathered} \dot\varepsilon^\circ + \varepsilon = \dot\sigma^\circ + \gamma\sigma^\circ \\ \dot\varepsilon^\circ + \varepsilon^\circ = \beta\dot\sigma^\circ + \sigma^\circ(\gamma + \beta - 1) + \sigma_*{}^\circ(1 - \beta) \\ \dot\varepsilon^\circ = \beta\dot\sigma^\circ \end{gathered} \tag{1.15}$$

In dimensionless form the loading on the initial section is

$$\begin{gathered} \sigma^\circ = 1 - \tau/\lambda, \quad 0 \leqslant \tau \leqslant \lambda \\ \sigma^\circ = 0, \quad \tau \geqslant \lambda \end{gathered}$$

The condition at the shock front for $x = \tau$

$$\sigma^\circ = -u^\circ$$

The circular superscripts, denoting dimensionless quantities, will be dropped in what follows.

The solution is determined by the assignment of values to the seven dimensional parameters σ_m, E_D, E_S, E_R, ρ_0, μ, θ. In the transition to the dimensionless parameters we have chosen, the number of determining parameters was reduced to three γ, β, λ. This is connected with the linearity of the equations determining the behavior of the medium. This reduction in the number of parameters makes it possible to apply the results of one computer calculation to a number of media and initial conditions.

For the numerical solution of the system (1.14), (1.15) the "cross" difference scheme [8] was employed. A scheme of this type has been applied in [9] for the description of one-dimensional motion of a gas and of an elastoplastic material. The solution is sought on a mesh that is uniform with respect to x and nonuniform with respect to τ. There are spatial and temporal layers of the mesh with integral and half-integral numbers. The quantity u is determined at points that have numbers that are half-integral for time and integral for space; the quantities σ and ε at points with numbers that are integral for time and half-integral for space. The difference equations have the form

$$\frac{u_j^{n+1/2} - u_j^{n-1/2}}{\Delta\tau^n} = \frac{\sigma_{j+1/2}^{\,n} - \sigma_{j-1/2}^{\,n} - q_{j+1/2}^{n-1/2} + q_{j-1/2}^{n+1/2}}{\Delta x} \tag{1.16}$$

$$\frac{\varepsilon_{j+1/2}^{n+1} - \varepsilon_{j+1/2}^{\,n}}{\Delta\tau^{n+1/2}} = \frac{u_{j+1}^{n+1/2} - u_j^{n+1/2}}{\Delta x} \tag{1.17}$$

$$\frac{\sigma_{j+1/2}^{n+1} - \sigma_{j+1/2}^{\,n}}{\Delta\tau^{n+1/2}} = a_1 \frac{\varepsilon_{j+1/2}^{n+1} - \varepsilon_{j+1/2}^{\,n}}{\Delta\tau^{n+1/2}} + a_2 \frac{\sigma_{j+1/2}^{n+1} + \sigma_{j+1/2}^{\,n}}{2} + a_3 \frac{\varepsilon_{j+1/2}^{n+1} + \varepsilon_{j+1/2}^{\,n}}{2} + a_4 \tag{1.18}$$

The equations of continuity and momentum are approximated in a way similar to that in [8, 9]. Equation (1.18) approximates the deformation law of the material (1.14), (1.15). The quantities a_1, a_2, a_3, a_4 are the coefficients associated with ε, σ, ε, and the free term in Eqs. (1.14)-(1.15) respectively. In (1.16) q is the Neumann–Richtmyer quadratic pseudoviscosity [10]

$$\begin{aligned} q_{j+1/2}^{n-1/2} &= \frac{C_q (u_{j+1}^{n-1/2} - u_j^{n-1/2})^2}{1 + (\varepsilon_{j+1/2}^{\,n} + \varepsilon_{j+1/2}^{n-1})/2} && \text{for } u_{j+1} < u_j, \\ q_{j+1/2}^{n-1/2} &= 0 && \text{for } u_{j+1} > u_j \end{aligned} \tag{1.19}$$

The presence of the pseudoviscosity in Eq. (1.16) causes a replacement of sharp shock fronts with zones of continuous variation of the parameters, values of the parameters at the boundaries of this zone being related by the jump conditions. Values of the constant C_q are bounded from above by the requirement that the width of the washing-out zone be small and from below by the requirement on smallness of perturbations arising because of the use of this particular calculational procedure, not through any inherent connection with the physics of the process. The value $C_q = 0.6$.

The system (1.16)-(1.18) approximates (1.14) and (1.15) to second-order accuracy. This follows from results of inserting Taylor series expansions of the unknown functions near the points (x_j, τ^n) and $(x_{j+1/2}, \tau^{n+1/2})$ into (1.16) and (1.17), respectively.

On each temporal layer sequential calculations are made of $\sigma_{1/2}^{n+1} = 1 - \tau^{n+1}/\lambda$, $\varepsilon_{1/2}^{n+1}$ by (1.18), $u_0^{n+1/2}$ by (1.17), then for $j \geq 1$ $u_j^{n+1/2}$ by (1.16), $\varepsilon_{j+1/2}^{n+1}$ by (1.17), $\sigma_{j+1/2}^{n+1}$ by (1.18), $q_{j+1/2}^{n+1/2}$ by (1.19), and finally $\Delta\tau^{n+3/2}$ and $\Delta\tau^{n+1} = (\Delta\tau^{n+1/2} + \Delta\tau^{n+3/2})/2$ by the stability condition, obtained by the method of freezing the coefficients [8].

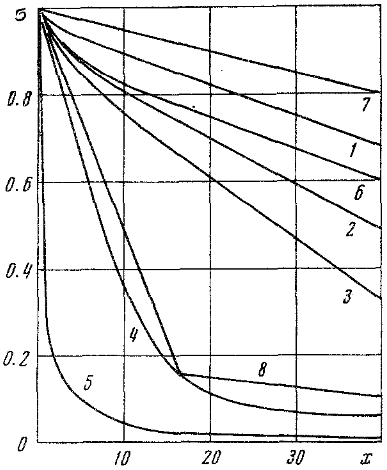

Fig. 2

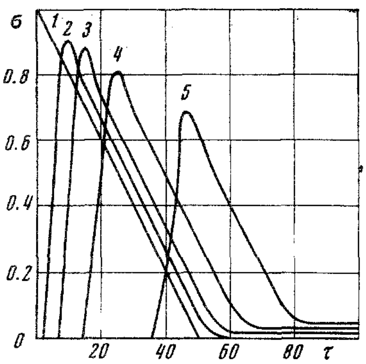

Fig. 3

In order to calculate the accuracy of the numerical method the same variant was worked out on meshes with steps Δx and $\Delta x/2$.

For small values of x the width of the washing-out zone was reduced to half when calculations were made with the finer mesh, while values of the stress and the velocity of the wave remained the same. For large x the width of the washing-out zone and the other parameters did not change. This indicates that the washing out of the discontinuity is the result of the model of the medium that takes viscosity properties into account, and not the result of the method of calculation involving the introduction of pseudoviscosity. In the solution of a similar problem by the method of characteristics, when the discontinuity was calculated accurately, the washing out of the latter in a medium having viscous properties was also obtained.

2. Six variants, differing in the values of λ, γ, and β, were calculated by computer:

Variant	γ	λ	β
1	1.1	50	0.5
2	2	50	0.5
3	4	50	0.5
4	2	5	0.5
5	2	0.5	0.5
6	2	50	1

Having this many variants one can analyze the influence of values of λ, γ, and β on the parameters of the wave.

As follows from calculation, a number of regions in the $x\tau$ plane emerge during the passage of the wave: 1) the undisturbed medium, 2) increase in stress and strain, 3) decrease in stress and increase in strain, 4) decrease in stress and strain.

The boundaries of the regions are: 1, 2) wave front, corresponding to the jump in σ and ε; 2, 3) maximum stress; 3, 4) maximum strain.

The configuration of the regions changes, depending on the values of λ, γ, and β.

Figure 2 shows plots of the dependence of the maximum stress on the distance. The numeration of the plots corresponds to the sequence of variants given above. A comparison of plots 1-3 shows that the waves are extinguished more rapidly with increasing γ. Physically an augmentation of γ corresponds to an increase in the deviation of the static compression diagram from the dynamical compression diagram, which is due to the more complete manifestation of the viscous properties of the medium.

It is advisable to make a comparison of plots 2-5 according to two interpretations, for μ=const or for θ=const. We put $\mu = 1000\ \text{sec}^{-1}$. Then λ is equal to 50, 5, and 0.5, corresponding to θ equal to 0.05, 0.005, and 0.0005 sec. From the plots it follows that with a decrease in θ the rate at which the stress attenuates with distance increases, that is, short waves die out more rapidly. We now hold θ constant. Let $\theta = 0.05$ sec. Then values of λ equal to 50, 5, and 0.5 correspond to values μ equal to 1000, 100, and 10 sec^{-1}. The variation in μ results in a variation in the scale of the spatial coordinate. From the condition $x/\mu = h/A$ it follows that the section $x = 10$ corresponds to one and the same value of h/A, equal to 10^{-3}, 10^{-2}, and 10^{-1} sec for the three values of μ. Taking this variation in scale into consideration, we obtain the result, on comparing plots 2, 4, 5, that with an increase in μ the waves die out more rapidly. Values μ=500-1000 sec^{-1} correspond to loose soils [3].

We shall discuss the time variation of σ, ε, u at fixed points of the medium. Figures 3-5 show plots of the time dependence of the dimensionless stress for $\lambda = 50$ and γ equal to 1.1, 2, and 4, respectively. In all cases the plots labeled with the numbers 1-5 refer to sections of the medium with coordinates 0, 5, 10, 20, and 40, respectively. Comparison of the plots shows that an increase in γ results in a slower rise of the stress to its maximum value and consequently to a more rapid conversion of the wave from a shock wave to a continuous one.

Figures 6 and 7 give plots of $\varepsilon(\tau)$ and $u(\tau)$ for $\lambda = 50$ and $\gamma = 2$ at the same sections. From a comparison of the plots for the stress and the strain (Figs. 4 and 6) it follows that maximum strain is attained dur-

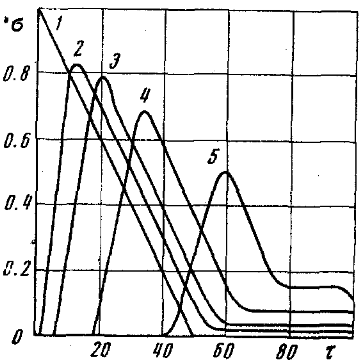

Fig. 4

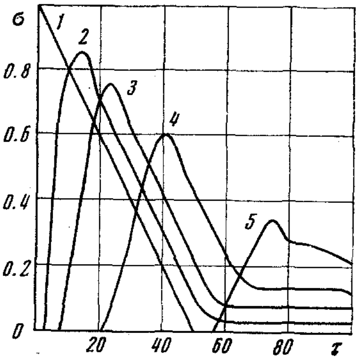

Fig. 5

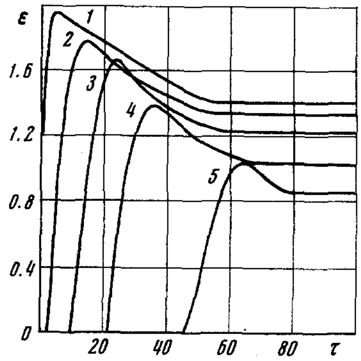

Fig. 6

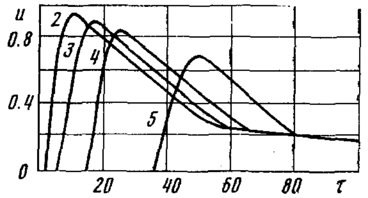

Fig. 7

ing the phase of diminishing stress. An increase in γ from 2 to 4 results in an increase in the maximum strain from 1.88 to 3.7. (The plot for $\gamma = 4$ is not given.) This is connected with the increase in the compressibility of the medium with an increase in γ.

3. We shall not compare the wave parameters in plastic and viscoplastic media. We use the model of a plastic medium, in which the loading occurs along the line $\sigma = E_D\varepsilon$ and the unloading along the line $\sigma - \sigma_m = E_R(\varepsilon - \varepsilon_m)$. This model is the limit case of the model applicable to a viscoplastic medium, corresponding to $E_S \to E_D$ or $\mu \to 0$. An analytic solution of the problem on the propagation of a plane wave in such a medium, produced by a loading whose variation in the initial section is given by the equations

$$\begin{gathered} \sigma = 0, \quad t \leqslant 0 \\ \sigma = \sigma_m (1 - t/\theta), \quad 0 \leqslant t \leqslant \theta \\ \sigma = 0, \quad t \geqslant \theta , \end{gathered} \tag{3.1}$$

has been obtained earlier [6].

In this case the maximum stress in the medium at the wavefront is given in terms of the variables h and t by the expressions

$$\frac{\sigma}{\sigma_m} = 1 - \frac{A_R^2 - A^2}{2A_R^2} \frac{h}{A\theta}, \quad \frac{h}{A\theta} \leqslant \frac{A_R}{A_R - A} \tag{3.2}$$

$$\frac{\sigma}{\sigma_m} = \frac{A_R - A}{A_R + A}\left(1 - \frac{(A_R - A)^2}{2A_R^2} \frac{h}{A\theta}\right), \quad \frac{A_R}{A_R - A} \leqslant \frac{h}{A\theta} \leqslant \frac{(A_R + A) A_R}{(A_R - A)^2}$$

$$\frac{\sigma}{\sigma_m} = \frac{(A_R - A)^2}{(A_R - A)^2}\left[1 - \frac{(A_R - A)^2 (A_R^2 - A^2)}{2AA_R (A_R + A)} \frac{h}{A\theta}\right], \quad \frac{h}{A\theta} \geqslant \frac{(A_R + A) A_R}{(A_R - A)^2} \tag{3.3}$$

$$A = (E_D\rho_0)^{1/2}, \quad A_R = (E_R\rho_0)^{1/2} \tag{3.4}$$

Figure 2 shows plots 7 and 8, constructed according to these equations for the case $A_R = \sqrt{2}A$, i.e., $\beta = 0.5$. Here plots 7 and 8 refer to the limit case ($\gamma \to 1$) of variants 2 and 4. The plot for a plastic medium, corresponding to variant 5, is not given, as it is practically identical with these plots. With a decrease in λ the plots for plastic and viscoplastic media approach each other.

Thus, calculations show that waves die out faster in viscoplastic media than they do in plastic media. This is connected with the fact that there are additional energy losses. Taking account of viscosity also results in a change in the wave profile – a conversion of the shock wave into a continuous compression wave. The maximum of the strain does not coincide with the maximum of the stress; it is attained during the phase of diminishing stress. The extent to which viscous properties influence the phenomena depends on the parameters λ, γ, and β. With an increase in λ and γ the influence of viscosity increases. For $\mu \to 0$ or for $E_D \to E_S$ the medium becomes plastic. A decrease in β corresponds to an increase in the influence of the plastic properties of the medium. It results in a more rapid extinction of waves both in plastic and in viscoplastic media. These results agree with experimental data.

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